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ON THE DIFFUSION OF HEAT IN A HOMOGENEOUS RECTANGULAR MASS, WITH SPECIAL REFERENCE TO BARS USED AS STANDARDS OF LENGTH.

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1. Considering the vast amount of time and labor which have been spent in developing the science and art of comparing standards of length, and the wonderful precision at present attainable in such work, it would seem somewhat surprising that more progress has not been made toward a complete theory of the behavior of metallic bars under varying temperature conditions. If we look over the scientific literature of the last half century, we shall find elaborate reports and memoirs on the appliances and methods adopted in determining coefficients of expansion and relative lengths of standards. But the chief object of all these appliances and methods has been to secure particular conditions, such as a steady low or high temperature of the bars under comparison; and the precise results for expansions and differences of length come out without reference to the intermediate states and effects arising from the slow diffusion of heat through the bars while changing from one steady temperature to another. Indeed, a dexterous avoidance of all the more complex phenomena presented by a cooling or heating bar has been the characteristic feature of experimental success. Our authorities on standards of length give us multitudes of facts and figures concerning the history of a bar during particular periods, but the history begins and ends abruptly and we are left to conjecture concerning the facts and figures of the more interesting periods.

The reason for this apparent anomaly, this paucity of information relative to the more recondite thermal properties of a bar, is obvious enough when we reflect that the present high grade of efficiency in the management of standards has only been attained after prolonged and diligent research. Obstacles quite independent of the standards themselves, of the most perplexing and baffling character, have had to be overcome. It is but natural and logical, therefore, that experimenters should have sought to avoid complexities as far as possible and direct their efforts so as to secure, in practical work particularly, the requisite precision at the minimum cost.

But precision is merely relative. What satisfies us to-day is almost sure to be unsatisfactory to-morrow. The perfection of methods and appliances which enabled us to overcome old difficulties has also revealed new ones, difficulties intimately related to, if not directly dependent on, the very states and conditions we have hitherto found it convenient to avoid or ignore. At present, precision of comparison appears to have reached a superior limit along the old lines of research; and future progress seems possible only by an advance in our knowledge of the more intricate thermal properties of metals. Apparently we must recur to the extraordinary studies of Fourier and Poisson; we must extend their work and adapt it to the practical needs of the computer; and we must learn how to determine readily for any standard the constants which define its intrinsic thermal relations and the relations it sustains to media in which it may be placed.

2. It is proposed to consider in the following pages the problem of a rectangular bar cooling or heating in a medium whose temperature remains sensibly constant. This is probably the easiest of the problems presented in the work of comparing standards of length, but it appears to be also the one whose solution will be most useful in a practical way.

Much, if not most, of the preliminary work has been done by Fourier* and Poisson†, and the reader is referred to their treatises for all elementary details. In his Chapter VII Fourier has considered the case of a rectangular prism of infinite length, subjected to a constant temperature at one end, and exposed to a uniform current of air at zero temperature; and in his Chapter VIII he has given a pretty full treatment of the case of a cube cooling in air or any other medium of sensibly constant temperature. Poisson in his Chapter IX has discussed at great length the case of the distribution of heat in a bar whose transverse dimensions are very small, and in his Chapter XI he has briefly considered the case of a rectangular mass of unlimited dimensions. Neither of these writers, however, appears to have had any practical applications in view, and elaborate and elegant as their work is, there remains much to be done to render it available for numerical applications.

We shall assume that initially the bar has a uniform excess in temperature over that of the surrounding medium; that the conductivity and thermal capacity of the bar and the emissivity are constant within the range of temperature considered.

Let the length, breadth, and thickness of the bar be denoted by $2a$, $2b$, and $2c$, respectively; and let any point within or on its bounding surface be defined

* *Théorie Analytique de la Chaleur*. Paris, 1822. *Analytical Theory of Heat*, by Joseph Fourier, translated, with notes, by Alexander Freeman, Cambridge, 1878. Our references in the text are to Freeman's translation.

† *Théorie Mathématique de la Chaleur*. Paris, 1835.

in position by a system of rectangular co-ordinates, whose origin is at the center of the bar and whose axes of x, y, z are parallel respectively to a, b, c. Call u the excess in temperature over that of the surrounding medium of any point x, y, z at any time t after the initial epoch. Then the partial differential equation, connecting t, u, x, y, z, which obtains in all cases of the diffusion of heat, is*

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right). \tag{1}$$

In this, k is the conductivity of the bar divided by its thermal capacity per unit of volume.

In addition to the relation (1) pertaining to any element of the bar at any time t, there are three other equations which express the equalities between the heat conducted across any element of the faces of the bar and that carried away by the surrounding medium. If we call the conductivity K and the emissivity H, these relations are

$$\mp K\frac{\partial u}{\partial x} = Hu \text{ for } x = \pm a, \tag{2}$$

$$\mp K\frac{\partial u}{\partial y} = Hu \text{ for } y = \pm b, \tag{3}$$

$$\mp K\frac{\partial u}{\partial z} = Hu \text{ for } z = \pm c. \tag{4}$$

Finally, there is the initial relation expressing the fact that at the beginning of cooling all points of the bar have the same temperature. If u_0 be the initial uniform excess in temperature of the bar above that of the surrounding medium, we have

$$u = u_0 \text{ for } t = 0. \tag{5}$$

The analytical solution of the problem consists in finding a function

$$u = F(u_0, x, y, z, t),$$

which will satisfy all of the relations (1) . . . (5).

3. The complete integral of equation (1), for the present case, as shown by Fourier in his Chapter VIII, consists of the product of three infinite series and the initial excess of temperature u_0. We may express this integral thus:

$$u = u_0 V' V'' V''',$$

$$V' = \sum_{n=0}^{n=\infty} v_n', \qquad V'' = \sum_{n=0}^{n=\infty} v_n'', \qquad V''' = \sum_{n=0}^{n=\infty} v_n'''; \tag{6}$$

*See Fourier's Chapter II, Theorem IV.

wherein

$$v_n' = A_n e^{-kt\lambda_n^2} \cos \lambda_n x,$$
$$v_n'' = B_n e^{-kt\mu_n^2} \cos \mu_n y,$$
$$v_n''' = C_n e^{-kt\nu_n^2} \cos \nu_n z.$$

In these expressions A_n, B_n, C_n, λ_n, μ_n, ν_n are constants varying from term to term with the index n, which is zero or any positive integer. These constants must be such that (6) will satisfy (2), . . . (5) as well as (1).

For brevity, put $h = H/K$. Then, if we substitute u and its derivatives from (6) in (2), (3), and (4), there result

$$\begin{aligned} a\lambda_n \tan a\lambda_n &= ah = \xi, \text{ say}, \\ b\mu_n \tan b\mu_n &= bh = \eta, \text{ say}, \\ c\nu_n \tan c\nu_n &= ch = \zeta, \text{ say}. \end{aligned} \tag{7}$$

Since the values of ξ, η, and ζ are known, the equations just written will determine the constants λ_n, μ_n, ν_n. Each of these equations, it will be observed, has an infinite number of real roots, the successive values of which are to be used in the several terms of v_n', v_n'', v_n''' in the order defined by the index n.

The constants A_n, B_n, C_n are determined in the following manner: Equation (5) requires that when $t = 0$,

$$\sum_{n=0}^{n=\infty} v_n' = \sum_{n=0}^{n=\infty} v_n'' = \sum_{n=0}^{n=\infty} v_n''' = 1,$$

that is, considering v_n' alone,

$$1 = A_0 \cos \lambda_0 x + A_1 \cos \lambda_1 x + A_2 \cos \lambda_2 x + \ldots.$$

If we multiply both sides of this equation by $\cos \lambda_n x \, dx$ and integrate between the limits 0 and a, it will be found that*

$$A_n = \frac{4 \sin a\lambda_n}{2a\lambda_n + \sin 2a\lambda_n}. \tag{8}$$

Corresponding expressions for B_n and C_n may be written down by an obvious interchange of symbols.

4. It is seen, therefore, that the use of equations (6) as they stand, requires the roots (or a certain number of them) of the transcendental equations (7), these roots entering respectively into each of the three factors of the corresponding

*See Fourier, Chapter VII, p. 313.

term of v_n', v_n'', or v_n'''. Herein lies the chief obstacle to the application of (6), and the principal feature of our work consists in showing how this obstacle may be overcome.

We shall show first how the roots of (7) may be found, though our solution of the problem requires in any case only the smallest of these roots.

For brevity, let us write

$$\begin{aligned}\theta_n &= a\lambda_n,\\ \varphi_n &= b\mu_n,\\ \psi_n &= c\nu_n.\end{aligned} \tag{9}$$

Then (7) become

$$\begin{aligned}\theta_n \tan \theta_n &= \xi,\\ \varphi_n \tan \varphi_n &= \eta,\\ \psi_n \tan \psi_n &= \zeta.\end{aligned} \tag{10}$$

It will suffice to consider the first of these equations, since they are all of the same type.

$\xi = ah$ is of zero dimension as respects length*, and it may, in general, have any value between 0 and ∞.

In the case of an iron bar cooling in air, h may be about $\frac{1}{800}$† with the centimeter as unit of length; so that for a bar of iron one meter long, we should have approximately (roughly, perhaps) $\xi = \frac{1}{16}$. But for other media, h may be so large as to make ξ exceed unity when a is no greater than a half meter. In any event, ξ will exceed unity if a is sufficiently great. It becomes essential, therefore, to distinguish two cases; namely, that in which $\xi < 1$ and that in which $\xi > 1$.

First, consider the case wherein $\xi < 1$, with a view to expressing θ_n in a series of ascending powers of ξ. When $\xi = 0$‡, we must have $\theta_n = n\pi$; i. e. θ_n is either zero or a multiple of π. Differentiating the first of (10), there results

$$(\tan \theta_n + \theta_n \sec^2 \theta_n) \frac{d\theta_n}{d\xi} = 1. \tag{11}$$

This gives, for $\xi = 0$,

$$\begin{aligned}\frac{d\theta_n}{d\xi} &= \infty \text{ when } n = 0,\\ &= (n\pi)^{-1} \text{ when } n \overline{\overline{>}} 1.\end{aligned}$$

*See Fourier, Chapter II, p. 128.

†Sir Wm. Thomson gives (Encyclopedia Britannica, 9th Edition, Article Heat) $H = 0.0002$ for a polished copper surface, and $K = 0.16$ for iron. Assuming the above value of H to apply for iron as well as copper we have $h = \frac{1}{800}$.

‡The physical meaning of this condition is that there can be no escape of heat from the bar.

It appears, therefore, that we cannot apply Maclaurin's series to the development of the first value of θ_n. But if we substitute for tan θ_n its value in series, we have

$$\xi = \theta_n^2 + \tfrac{1}{3}\theta_n^4 + \tfrac{2}{15}\theta_n^6 + \ldots;$$

whence, by reversion, we get for the first roots of (10)

$$\begin{aligned} \theta_0^2 &= \xi - \tfrac{1}{3}\xi^2 + \tfrac{4}{45}\xi^3 - \tfrac{16}{945}\xi^4 + T_5,^* \\ \varphi_0^2 &= \eta - \tfrac{1}{3}\eta^2 + \tfrac{4}{45}\eta^3 - \tfrac{16}{945}\eta^4 + T_5, \\ \psi_0^2 &= \zeta - \tfrac{1}{3}\zeta^2 + \tfrac{4}{45}\zeta^3 - \tfrac{16}{945}\zeta^4 + T_5. \end{aligned} \tag{12}$$

These series will suffice for the calculation of the first values of θ_n, φ_n, ψ_n. The convergence is rapid for small values of ξ, η, ζ, and quite adequate even when they are as great as unity.

For the larger values of θ_n we may apply Maclaurin's series. Thus, by eliminating the trigonometric forms from (11), we get

$$\frac{d\theta_n}{d\xi} = \theta_n(\theta_n^2 + \xi + \xi^2)^{-1};$$

whence, by an application of the theorem of Leibnitz, we readily find that, for $\xi = 0$,

$$\begin{aligned} \frac{d\theta_n}{d\xi} &= +(n\pi)^{-1}, \\ \frac{d^2\theta_n}{d\xi^2} &= -2(n\pi)^{-3}, \\ \frac{d^3\theta_n}{d\xi^3} &= +12(n\pi)^{-5} - 2(n\pi)^{-3}, \\ &\ldots\ldots\ldots\ldots \end{aligned} \tag{13}$$

Hence, we write

$$\theta_n = n\pi\{1 + \xi(n\pi)^{-2} - \xi^2(n\pi)^{-4} + \xi^3[2(n\pi)^{-6} - \tfrac{1}{3}(n\pi)^{-4}] + T_4\}. \tag{14}$$
$$n \overline{\gtrless} 1, \quad \xi \overline{\gtrless} 1.$$

This series converges rapidly, especially for large values of n; it shows, in fact, that the larger the root the more nearly is it equal to $n\pi$.

Second, when $\xi > 1$, we may write the first of (10) in the form

$$\cot\theta_n = \theta_n\xi_1,$$

wherein
$$\xi_1 = \xi^{-1}. \tag{15}$$

Now the first of these equations gives for $\xi_1 = 0$,†

* We shall use throughout this paper T_n to indicate that the next term of the series beyond those given is a term of the nth order. Thus the next term in the first of (12) is of the 5th order with respect to the variable ξ.

† The physical meaning of this is that the heat is instantly dissipated on reaching the surface of the bar, or escapes "freely."

$$\begin{aligned}
\theta_n &= \tfrac{1}{2}(2n+1)\pi = \vartheta, \text{ say,} \\
\frac{d\theta_n}{d\xi_1} &= -\vartheta, \\
\frac{d^2\theta_n}{d\xi_1^2} &= +2\vartheta, \\
\frac{d^3\theta_n}{d\xi_1^3} &= -(6-2\vartheta^2)\vartheta, \\
&\ldots\ldots\ldots
\end{aligned} \tag{16}$$

Therefore, in accordance with Maclaurin's series,

$$\theta_n = \vartheta[1 - \xi^{-1} + \xi^{-2} - (1 - \tfrac{1}{3}\vartheta^2)\xi^{-3} + T_4]. \tag{17}$$
$$\xi \overline{\overline{>}} 1, \quad \vartheta = \tfrac{1}{2}(2n+1)\pi.$$

Except for large values of ξ, this series converges too slowly to be of much practical value. Our process, however, does not require its use. We give it only to facilitate a numerical verification of the equivalence of the original and transformed expressions for V', V'', V'''. See § 7.

5. It will appear in the sequel that, for small values of ξ, η, and ζ, the series of (6) converge rapidly toward their first terms with the lapse of time from the initial epoch. On this account, it is desirable to have a ready method of computing v_0', v_0'', and v_0'''. The factors in these terms are all in convenient form except A_0, B_0, and C_0, which are implicit functions of ξ, η, and ζ, respectively, as defined by (8), (9), and (10). They may be expressed directly in terms of ξ, η, ζ in the following manner. Taking A_0 as the type, (8) and (9) give

$$A_0 = \frac{4 \sin \theta_0}{2\theta_0 + \sin 2\theta_0}.$$

Substituting for $\sin \theta_0$ and $\sin 2\theta_0$ their values in series, we find

$$A_0 = 1 + \tfrac{1}{6}\theta_0^2 - \tfrac{1}{360}\theta_0^4 + T_6.$$

Replacing θ_0^2 in this by its value in (12), there result

$$\begin{aligned}
A_0 &= 1 + \tfrac{1}{6}\xi - \tfrac{7}{120}\xi^2 + T_3, \\
B_0 &= 1 + \tfrac{1}{6}\eta - \tfrac{7}{120}\eta^2 + T_3, \\
C_0 &= 1 + \tfrac{1}{6}\zeta - \tfrac{7}{120}\zeta^2 + T_3.
\end{aligned} \tag{18}$$

From (6) and (9), we have

$$\begin{aligned}
v_0' &= A_0 e^{-kt(\theta_0/a)^2} \cos \theta_0 \frac{x}{a}, \\
v_0'' &= B_0 e^{-kt(\phi_0/b)^2} \cos \varphi_0 \frac{y}{b}, \\
v_0''' &= C_0 e^{-kt(\psi_0/c)^2} \cos \psi_0 \frac{z}{c}.
\end{aligned} \tag{19}$$

Computation of numerical values from these expressions is now easy, since θ_0, φ_0, ψ_0 are given by (12) and A_0, B_0, C_0 by (18).

In addition to v_0', v_0'', and v_0''' we need

$$\sum_{n=1}^{n=\infty} v_n', \quad \sum_{n=1}^{n=\infty} v_n'', \text{ and } \sum_{n=1}^{n=\infty} v_n'''.$$

These we proceed to expand as functions of ξ, η, and ζ, respectively, by Maclaurin's series.

By reference to (8), (9), and (14) it is seen that when $n \geqq 1$ and $\xi = 0$,

$$A_n = 0,$$
$$e^{-kt\lambda_n^2} = e^{-kt(n\pi/a)^2},$$
$$\cos\lambda_n x = \cos n\pi\frac{x}{a};$$
$$\frac{dA_n}{d\xi} = -2(-1)^{n+1}(n\pi)^{-2},$$
$$\frac{d^2A_n}{d\xi^2} = +12(-1)^{n+1}(n\pi)^{-4},$$
$$\cdots\cdots\cdots\cdots;$$

and
$$\frac{d\lambda_n}{d\xi} = +(an\pi)^{-1},$$
$$\cdots\cdots\cdots\cdots$$

Hence, when $\xi = 0$,

$$\frac{dv_n'}{d\xi} = -\frac{2(-1)^{n+1}}{\pi^2 n^2} e^{-kt(n\pi/a)^2}\cos n\pi\frac{x}{a},$$
$$\frac{d^2v_n'}{d\xi^2} = +\frac{12(-1)^{n+1}}{\pi^4 n^2} e^{-kt(n\pi/a)^2}\cos n\pi\frac{x}{a}$$
$$+\frac{8kt(-1)^{n+1}}{a^2\pi^2n^2} e^{-kt(n\pi/a)^2}\cos n\pi\frac{x}{a}$$
$$+\frac{4x(-1)^{n+1}}{a\pi^3n^3} e^{-kt(n\pi/a)^2}\sin n\pi\frac{x}{a},$$
$$\cdots\cdots\cdots\cdots\cdots$$

If now we write for brevity,

$$\left.\begin{aligned} S_1' &= \frac{2}{\pi^2}\sum_{n=1}^{n=\infty}\frac{(-1)^{n+1}}{n^2}e^{-kt(n\pi/a)^2}\cos n\pi\frac{x}{a}, \\ S_2' &= \frac{2}{\pi^4}\sum_{n=1}^{n=\infty}\frac{(-1)^{n+1}}{n^4}e^{-kt(n\pi/a)^2}\cos n\pi\frac{x}{a}, \\ &\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \quad (20)$$

we have for $\xi = 0$,

$$\frac{\partial \sum\limits_{n=1}^{n=\infty}{}' v_n'}{\partial \xi} = - S_1',$$

$$\frac{\partial^2 \sum\limits_{n=1}^{n=\infty}{}' v_n'}{\partial \xi^2} = + 2 \left(3S_2' - t\frac{\partial S_2'}{\partial t} - x\frac{\partial S_2'}{\partial x} \right),$$

. . . .

Finally, if we denote by S_1'', S_2'', etc., and by S_1''', S_2''', etc., the series corresponding to (20) when a and x are replaced by b and y, and c and z, respectively, there result

$$\begin{aligned} V' &= v_0' - S_1'\xi + \left(3S_2' - t\frac{\partial S_2'}{\partial t} - x\frac{\partial S_2'}{\partial x} \right) \xi^2 + T_3, \\ V'' &= v_0'' - S_1''\eta + \left(3S_2'' - t\frac{\partial S_2''}{\partial t} - y\frac{\partial S_2''}{\partial y} \right) \eta^2 + T_3, \\ V''' &= v_0''' - S_1'''\zeta + \left(3S_2''' - t\frac{\partial S_2'''}{\partial t} - z\frac{\partial S_2'''}{\partial z} \right) \zeta^2 + T_3. \end{aligned} \tag{21}$$

Equations (12), and (18) to (21) afford a complete solution of the problem for the case wherein ξ, η, ζ are each less than unity. The series in (20) converge with such rapidity that for any but the earliest stages of cooling their first terms will suffice. For practical applications to a standard of length of ordinary dimensions, cooling or heating in air, we may neglect all terms after the first in (21).

The values of S_1', etc., in (20) might be expressed by a series of definite integrals similar to that derived in §8, but it does not appear that any practical advantage would be gained thereby.

6. Considering still the case in which ξ, η, and ζ are each less than unity, it will be of interest to have expressions for certain average temperatures, namely: (α) the average temperature of the whole mass of the bar, (β) the average temperature of any face of the bar, and (γ) the average temperature along its longitudinal axis.

(α). The average temperature of the whole mass is defined by the definite integral

$$(8abc)^{-1} \int_{-a}^{+a} u dx \int_{-b}^{+b} dy \int_{-c}^{+c} dz.$$

A glance at (20) and (21) shows that this is equal to

$$u_0(8abc)^{-1}\int_{-a}^{+a} v_0' dx \int_{-b}^{+b} v_0'' dy \int_{-c}^{+c} v_0''' dz + T_2;$$

whence by reference to (19) we get the following result:*

$$(\alpha) = u_0 A_0 B_0 C_0 \frac{\sin\theta_0 \sin\varphi_0 \sin\psi_0}{\theta_0\, \varphi_0\, \psi_0} e^{-kt(\lambda_0^2+\mu_0^2+\nu_0^2)} + T_2,$$

which, by reason of (12) and (18), is easily reduced to

$$(\alpha) = u_0 e^{-kt(\lambda_0^2+\mu_0^2+\nu_0^2)} + T_2. \tag{22}$$

(β). The average temperature of any face of the bar, for example the face $2a \times 2b$, is expressed thus:

$$\sigma_c'''(4ab)^{-1}\int_{-a}^{+a} u dx \int_{-b}^{+b} dy,$$

wherein σ_c''' stands for the second member of the last of equations (21) when $z = c$. Integrating, there results

$$(\beta) = u_0\ \sigma_c'''(e^{-kt(\lambda_0^2+\mu_0^2)} + T_2). \tag{23}$$

It is easily seen that this differs from (22) by a term of the first order in ζ.

(γ). In a similar manner we find for the average temperature along the longitudinal axis of the bar,

$$(\gamma) = u_0\ \sigma_0''\ \sigma_0'''(e^{-kt\lambda_0^2} + T_2), \tag{24}$$

in which σ_0'' and σ_0''' stand respectively for the second and third of (21) when $y = 0$ and $z = 0$. Obviously, (24) differs from (22) and (23) by terms of the first order in η and ζ.

7. Returning now to the case in which ξ, η, and ζ are each greater than unity, we proceed to show how the sums in (6) may be expanded in series of ascending powers of ξ^{-1}, η^{-1}, and ζ^{-1}. Referring to (15) and (16) and using for brevity α, β, γ as defined below, we get for $\xi^{-1} = \xi_1 = 0$,

$$\alpha = A_n = +2(-1)^n\vartheta^{-1},$$
$$\frac{\partial\alpha}{\partial\xi_1} = 0,$$
$$\frac{\partial^2\alpha}{\partial\xi_1^2} = -2(-1)^n\vartheta,$$
$$\cdots\cdots\cdots\cdots;$$

*Compare Fourier, Chapter VIII, p. 328, for the case in which $a = b = c$.

$$\beta = e^{-kt\lambda_n^2} = + e^{-kt(\vartheta/a)^2},$$

$$\frac{\partial\beta}{\partial\xi_1} = + 2kt(\vartheta/a)^2 e^{-kt(\vartheta/a)^2},$$

$$\frac{\partial^2\beta}{\partial\xi_1^2} = -\left[6kt(\vartheta/a)^2 - (2kt)^2(\vartheta/a)^4\right]e^{-kt(\vartheta/a)^2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\gamma = \cos\lambda_n x = + \cos\vartheta\frac{x}{a},$$

$$\frac{\partial\gamma}{\partial\xi_1} = + \vartheta\frac{x}{a}\sin,\vartheta\frac{x}{a},$$

$$\frac{\partial^2\gamma}{\partial\xi_1^2} = -\vartheta^2\left(\frac{x}{a}\right)^2\cos\vartheta\frac{x}{a} - 2\vartheta\frac{x}{a}\sin\vartheta\frac{x}{a},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

If now we put V_0' for V' when $\xi^{-1} = \xi_1 = 0$, the preceding relations enable us to derive readily the following:

$$\left.\begin{aligned}
V_0' &= \frac{4}{\pi}\sum_{n=0}^{n=\infty}\frac{(-1)^n}{2n+1}e^{-kt(2n+1)^2(\pi/2a)^2}\cos\tfrac{1}{2}(2n+1)\pi\frac{x}{a},\\
\frac{\partial V'}{\partial\xi_1} &= -\left(2t\frac{\partial V_0'}{\partial t} + x\frac{\partial V_0'}{\partial x}\right),\\
\frac{\partial^2 V'}{\partial\xi_1^2} &= +\left\{\begin{aligned}&\left(\frac{a^2}{k}+6t\right)\frac{\partial V_0'}{\partial t} + 4t^2\frac{\partial^2 V_0'}{\partial t^2}\\ &+ 2x\frac{\partial V_0'}{\partial x} + x^2\frac{\partial^2 V_0'}{\partial x^2} + 4tx\frac{\partial^2 V_0'}{\partial t\,\partial x}\end{aligned}\right\}.
\end{aligned}\right\} \tag{25}$$

Hence by Maclaurin's series

$$\left.\begin{aligned}
V' &= V_0' + \frac{\partial V'}{\partial\xi^{-1}}\xi^{-1} + \ldots,\\
V'' &= V_0'' + \frac{\partial V''}{\partial\eta^{-1}}\eta^{-1} + \ldots,\\
V''' &= V_0''' + \frac{\partial V'''}{\partial\zeta^{-1}}\zeta^{-1} + \ldots;
\end{aligned}\right\} \tag{26}$$

in which V_0'', V_0''' and the derivatives of V'', V''' with respect to η^{-1} and ζ^{-1} are obtained from (25) by replacing a, x by b, y and c, z respectively.

The applicability of (26) for numerical computations will depend chiefly on the rapidity of convergence of the series expressing V_0', V_0'', and V_0'''. These series (see the first of (25)) converge rapidly when the time is sufficiently great, but rather slowly during the earlier stages of cooling. They may be expressed, however, by a series of definite integrals which converge with extreme sud-

denness for all but large values of the time; and as the process of transformation from the former to the latter series is of great utility in other problems as well as in this, we shall give the steps of its present application rather fully in the next section.

8. If in the equation*

$$e^{-(\beta/2\alpha)^2} = \frac{2\alpha}{\sqrt{\pi}} \int_0^\infty e^{-\alpha^2\gamma^2} d\gamma \cos \beta\gamma$$

we put

$$4\alpha^2 = \frac{1}{kt}$$

and

$$\beta = (2n+1)\frac{\pi}{2a},$$

the value of V_0' in (25) becomes

$$V_0' = \frac{4}{\pi\sqrt{(\pi kt)}} \int_0^\infty e^{-\gamma^2/4kt} d\gamma \sum_{n=0}^{n=\infty} \frac{(-1)^n}{2n+1} \cos \tfrac{1}{2}(2n+1)\pi\frac{x}{a} \cos \tfrac{1}{2}(2n+1)\pi\frac{\gamma}{a}$$

$$= \frac{2}{\pi\sqrt{(\pi kt)}} \int_0^\infty e^{-\gamma^2/4kt} d\gamma \sum_{n=0}^{n=\infty} \frac{(-1)^n}{2n+1} \left\{ \begin{array}{l} \cos \tfrac{1}{2}(2n+1)\pi\left(\dfrac{\gamma+x}{a}\right) \\ + \cos \tfrac{1}{2}(2n+1)\pi\left(\dfrac{\gamma-x}{a}\right) \end{array} \right\} . \quad (27)$$

Now the series under the sign Σ in this equation is periodically equal to zero and $\frac{1}{2}\pi$, as γ varies over its range from 0 to ∞; and hence V_0' consists of a definite integral whose successive parts are multiplied alternately by 0 and $\frac{1}{2}\pi$. These facts may be shown in the following manner:

Take the difference between the well known equation

$$\frac{\frac{1}{2}(1-g^2)}{1-2g\cos p+g^2} = \tfrac{1}{2} + g\cos p + g^2\cos 2p + g^3\cos 3p + \ldots$$

and the equation which follows from the substitution of $\pi + p$ for p. The result is

$$\tfrac{1}{4}\left[\frac{1-g^2}{1-2g\cos p+g^2} - \frac{1-g^2}{1+2g\cos p+g^2}\right] = g\cos p + g^3\cos 3p + \ldots.$$

For p in this, substitute $90^\circ - (G+p)$. We find

$$\tfrac{1}{4}\left[\frac{1-g^2}{1-2g\sin(G+p)+g^2} - \frac{1-g^2}{1+2g\sin(G+p)+g^2}\right] =$$

$$g\sin(G+p) - g^3\sin 3(G+p) + \ldots.$$

* See Vol. III, p. 78, of the ANNALS OF MATHEMATICS.

Multiply this last by dp and integrate between the limits $-P$ and $\pi+P$. The result is

$$\tfrac{1}{4}\int_{-P}^{\pi+P}\left[\frac{1-g^2}{1-2g\sin(G+p)+g^2}-\frac{1-g^2}{1+2g\sin(G+p)+g^2}\right]dp=Q, \text{ say.} \quad (28)$$

But Q is evidently identical with the quantity under the sign Σ in (27) when

$$\left.\begin{aligned} g&=1,\\ G&=\frac{\pi\gamma}{2a},\\ P&=\frac{\pi x}{2a}. \end{aligned}\right\} \quad (29)$$

The element-functions of the integrals in the first member of (28) are obviously zero for $g=1$ except when $G+p$ is $\frac{1}{2}(4i+1)\pi$ in the first and $\frac{1}{2}(4i+3)\pi$ in the second, i being zero or any integer. In these exceptional cases the element-functions are infinite; and the values of the integrals depend on the limits $-P$ and $P+\pi$ only so far as they determine within what range γ can give to $G+p$ the above named multiples of $\frac{1}{2}\pi$. Hence, in evaluating either of the integrals, as for example the first, for any case in which $\sin(G+p)=+1$, we may enlarge the limits to $-\infty$ and $+\infty$ without affecting the result. This premised, the required evaluation may be accomplished by a process frequently employed by Poisson in the treatment of similar integrals.*

Substitute for g, $1-w$, where w is a small quantity approximating zero as g approaches unity. Also, replace $G+p$ by $\frac{1}{2}(4i+1)\pi+q$ in the first integral and by $\frac{1}{2}(4i+3)\pi+q$ in the second, q being, like w, a small quantity. Then, neglecting terms of the third and higher orders in q, either of the integrals becomes

$$\tfrac{1}{2}\int_{-q_1}^{+q_2}\frac{w\,dq}{w^2+q^2},$$

wherein q_1 and q_2, as seen above, may have any values greater than an infinitesimal. Hence the value of either integral is

$$\tfrac{1}{2}\left(\arc\tan\frac{+q_2}{w}-\arc\tan\frac{-q_1}{w}\right)=\tfrac{1}{2}\pi$$

when $w=0$.

*See, for instance, Chapters VII and VIII of the *Théorie Mathématique de la Chaleur*.

We must now find the limits of G, or rather γ as defined by the second of (29), within which the factor Q is $\frac{1}{2}\pi$. For the first integral the lower limits of γ are given by the equation

$$G=\tfrac{1}{2}(4i+1)\pi-\pi\quad-P,$$

or

$$\gamma=\ (4i+1)a-2a-x;$$

whence

$$\begin{aligned}\gamma&=-(\ a+x)\ \text{for } i=0,\\ &=+(3a-x)\ \text{for } i=1,\\ &=+(7a-x)\ \text{for } i=2,\\ &\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots;\end{aligned}$$

and the upper limits are given by the equation

$$\gamma=(4i+1)a+x,$$

whence

$$\begin{aligned}\gamma&=\ a+x\ \text{for } i=0,\\ &=5a+x\ \text{for } i=1,\\ &=9a+x\ \text{for } i=2,\\ &\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

Similarly, the lower and upper limits for the second integral are, respectively,

$$\begin{aligned}a-x \text{ and } 3a+x \text{ for } i=0,\\ 5a-x \text{ and } 7a+x \text{ for } i=1,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

Hence, bearing in mind that γ is positive only, we find for the transformed expression of (27)

$$\begin{aligned}V_0'&=\frac{1}{\sqrt{(\pi kt)}}\left[\int_0^{a+x} e^{-\gamma^2/4kt}\,d\gamma-\int_{a-x}^{3a+x} e^{-\gamma^2/4kt}\,d\gamma+\int_{3a-x}^{5a+x} e^{-\gamma^2/4kt}\,d\gamma-\ldots\right]\\ &=\frac{1}{\sqrt{(\pi kt)}}\left[\int_0^{a-x} e^{-\gamma^2/4kt}\,d\gamma-\int_{a+x}^{3a-x} e^{-\gamma^2/4kt}\,d\gamma+\int_{3a+x}^{5a-x} e^{-\gamma^2/4kt}\,d\gamma-\ldots\right].\end{aligned}\tag{30}$$

If now we write for brevity

$$\begin{aligned}r_0&=a/2\sqrt{(kt)},\\ r&=x/2\sqrt{(kt)},\end{aligned}\tag{31}$$

and replace γ by $2\gamma\sqrt{(kt)}$, there results

$$V_0'=\frac{2}{\sqrt{\pi}}\left[\int_0^{r_0-r} e^{-\gamma^2}d\gamma-\int_{r_0+r}^{3r_0-r} e^{-\gamma^2}d\gamma+\ldots\right].\tag{32}$$

The corresponding expressions for V_0'' and V_0''' are obtained by merely interchanging a, x with b, y and c, z.

It will be observed, of course, that the series of definite integrals in (30) or (32) possesses all the properties and requirements of the less complex series in the first of (25). We shall not trace out these details, but the reader's attention may be drawn to the remarkable way in which (30) and (32) become zero for $x = \pm a$ and unity for $t = 0$.

In case ξ, η, ζ are so large that we may neglect all terms after the first in (26),

$$u = u_0 V_0' V_0'' V_0'''.$$

This is always an approximate expression for the temperature when ξ, η, and ζ are greater than unity; and, other relations remaining constant, its degree of approximation increases as the dimensions a, b, c increase. For points near the surface of a very large mass, comparable in dimensions and other circumstances with the earth, say, we may write

$$u = 8u_0(\pi)^{-\frac{3}{2}} \int_0^{(a-x)/2\sqrt{(kt)}} e^{-\gamma^2} d\gamma \int_0^{(b-y)/2\sqrt{(kt)}} e^{-\gamma^2} d\gamma \int_0^{(c-z)/2\sqrt{(kt)}} e^{-\gamma^2} d\gamma. \tag{33}$$

For points near the surface of a slab of infinite dimensions in the directions of b and c, and finite but great dimension in the direction a, the last equation gives

$$u = \frac{2u_0}{\sqrt{\pi}} \int_0^{(a-x)/2\sqrt{(kt)}} e^{-\gamma^2} d\gamma,$$

a well known result.

The derivatives of V_0' with respect to x and t which enter the last two of equations (25), are easily found from (32). For brevity, write

$$\begin{aligned}
\rho_1 &= 2nr_0 + r_0 - r, \\
\rho_2 &= 2nr_0 + r_0 + r, \\
R_1 &= \sum_{n=0}^{n=\infty} (-1)^n \left[e^{-\rho_1^2} - e^{-\rho_2^2}\right], \\
R_2 &= \sum_{n=0}^{n=\infty} (-1)^n \left[\rho_1 e^{-\rho_1^2} + \rho_2 e^{-\rho_2^2}\right], \\
R_3 &= \sum_{n=0}^{n=\infty} (-1)^n \left[\rho_1^2 e^{-\rho_1^2} - \rho_2^2 e^{-\rho_2^2}\right], \\
R_4 &= \sum_{n=0}^{n=\infty} (-1)^n \left[\rho_1^3 e^{-\rho_1^2} + \rho_2^3 e^{-\rho_2^2}\right].
\end{aligned} \tag{34}$$

Then we have

$$
\left.
\begin{aligned}
\frac{\partial V_0'}{\partial x} &= -\frac{R_1}{\sqrt{(\pi k t)}},\\
\frac{\partial^2 V_0'}{\partial x^2} &= -\frac{R_2}{kt\sqrt{\pi}},\\
\frac{\partial V_0'}{\partial t} &= -\frac{R_2}{t\sqrt{\pi}},\\
\frac{\partial^2 V_0'}{\partial t^2} &= +\frac{1}{t^2\sqrt{\pi}}\left(\tfrac{3}{2}R_2 - R_4\right),\\
\frac{\partial^2 V_0'}{\partial x \partial t} &= +\frac{1}{2t\sqrt{(\pi k t)}}\left(R_1 - 2R_3\right).
\end{aligned}
\right\} \qquad (35)
$$

9. We are prepared now to derive any species of average value of the temperature for the case under consideration. Of these, two may suffice, namely: (α) the average temperature of the whole mass and (β) the average temperature over any face of the mass as $2a \times 2b$.

(α). For brevity put

$$
\begin{aligned}
M' &= (2a)^{-1}\int_{-a}^{+a} V' dx,\\
M'' &= (2b)^{-1}\int_{-b}^{+b} V'' dy,\\
M''' &= (2c)^{-1}\int_{-c}^{+c} V''' dz.
\end{aligned}
$$

Then

$$(\alpha) = u_0 M' M'' M'''. \qquad (36)$$

If now we write

$$
\left.
\begin{aligned}
N &= (2a)^{-1}\int_{-a}^{+a} V_0' dx,\\
N_1 &= (2a)^{-1}\int_{-a}^{+a} \frac{\partial V_0'}{\partial \xi^{-1}} dx,
\end{aligned}
\right\} \qquad (37)
$$

it appears from (25) and (26) that

$$M' = N + N_1 \xi^{-1} + T_2. \qquad (38)$$

Similar expressions obtain, of course, for M'' and M'''.

It remains to evaluate the integrals in (37). They are simple in form but present some of the difficulties peculiar to the class of functions to which they belong. The first of them is readily evaluated by direct operation on the first of (25), while the second comes by a more tedious process through the intervention of the expressions in (31), (34), and (35). The results are

$$N = 8\pi^{-2} \sum_{n=0}^{n=\infty} \frac{1}{(2n+1)^2} e^{-kt(2n+1)^2\pi^2/4a^2},$$

$$N_1 = \pi^{-\frac{1}{2}} \sum_{n=0}^{n=\infty} (-1)^n \left[r_0^{-1}\left(e^{-4n^2r_0^2} - e^{-4(n+1)^2r_0^2}\right) + 2(2n+1)\int_{2nr_0}^{(2n+1)r_0} e^{-\rho^2} d\rho \right]. \tag{39}$$

The steps in the derivation of N_1 are briefly these:

$$(2a)^{-1}\int_{-a}^{+a} \frac{\partial V'}{\partial \xi^{-1}} dx = \frac{1}{a\sqrt{\pi}} \int_{-a}^{+a} \left(R_2 + \frac{1}{2\sqrt{(kt)}} R_1 x\right) dx$$

$$= \frac{2\sqrt{(kt)}}{a\sqrt{\pi}} \int_{-r_0}^{+r_0} \left[\frac{\partial R_1}{\partial r} + R_1 r\right] dr = r_0^{-1}\pi^{-\frac{1}{2}} \left\{ [R_1]_{-r_0}^{+r_0} + \int_{-r_0}^{+r_0} R_1 r dr \right\}$$

$$= r_0^{-1}\pi^{-\frac{1}{2}} \left\{ [R_1]_{-r_0}^{+r_0} + 2 \sum_{n=0}^{n=\infty} (-1)^n \int_{2nr_0}^{2(n+1)r_0} e^{-\rho^2}(2nr_0 + r_0 - \rho) d\rho \right\}$$

$$= r_0^{-1}\pi^{-\frac{1}{2}} \sum_{n=0}^{n=\infty} (-1)^n \left\{ \left[e^{-\rho^2}\right]_{2(n+1)r_0}^{2nr_0} + 2(2n+1) r_0 \int_{2nr_0}^{2(n+1)r_0} e^{-\rho^2} d\rho \right\},$$

whence the expression in the text.

The second members of (39) though complex in form are very easily evaluated, by reason of the rapid convergence of the series on which they depend. It may be noted that for $t=0$, $N=1$ and $N_1=0$, and that for $t=\infty$, $N=N_1=0$, as required by the conditions of the problem.

(β). To get the average temperature of the face $2a \times 2b$, we have only to multiply $u_0 M' M''$ by the value of V''' in (26) when $z=c$. Calling this value V_c''' we have

$$(\beta) = u_0 M' M'' V_c'''. \tag{40}$$

10. It is believed that the solutions given by (6), (21), and (26) will answer most of the requirements in practical applications to standards of length of rect-

angular form. It is possible, however, that the developments with respect to ξ, η, and ζ may prove to be in some cases insufficiently convergent. This can happen only when ξ, η, and ζ are near unity, and the obvious modification requisite for the treatment of such cases, is a development with respect to $1-\xi$, $1-\eta$, and $1-\zeta$. But in the absence of a definite knowledge of the circumstances giving rise to this phase of the problem, it does not appear useful to do more than suggest the modification.

11. For the purpose of illustrating the application of the preceding theory, we shall work out in detail some numerical examples.

First, take the case of an iron bar, $1^m = 100^{cm}$ long and $4^{cm} \times 4^{cm}$ in cross section; cooling in air. We have for iron, according to Sir W. Thomson,* in C. G. S. units, $k = 0.185$, and we may, as in §4, assume $h = \frac{1}{800}$. Then, since $a = 50^{cm}$, $b = 2^{cm}$, and $c = 2^{cm}$,

$$\xi = \tfrac{1}{16}, \quad \eta = \zeta = \tfrac{1}{400}.$$

With these data, equations (12) give

$$\begin{aligned} \log\, \theta_0^2 &= 8.78689 - 10, \\ \log\, \varphi_0^2 &= 7.39758 - 10, \\ \log\, \psi_0^2 &= 7.39758 - 10. \end{aligned}$$

Also

$$\begin{aligned} \theta_0 &= 14^\circ\ 10'\ 35'', \\ \varphi_0 &= \ \ 2\ \ \ 51\ \ \ 49, \\ \psi_0 &= \ \ 2\ \ \ 51\ \ \ 49; \end{aligned}$$

and

$$\begin{aligned} \log k \left(\frac{\theta_0}{a}\right)^2 &= 4.65621 - 10, \\ \log k \left(\frac{\varphi_0}{b}\right)^2 &= 6.06269 - 10, \\ \log k \left(\frac{\psi_0}{c}\right)^2 &= 6.06269 - 10. \end{aligned}$$

From (18) we find

$$\begin{aligned} \log A_0 &= 0.00440, \\ \log B_0 &= 0.00018, \\ \log C_0 &= 0.00018. \end{aligned}$$

Encyclopedia Britannica, 9th Ed., article Heat.

We are now prepared to compute the factors V', V'', and V''' of (21) for any values of t, x, y, z. For example, let us determine these factors and hence the temperatures for several points along the axis of the bar at the end of twenty minutes from the initial epoch. That is, make $t = 1200^s$; x successively 0, $\frac{1}{4}a$, $\frac{1}{2}a$, $\frac{3}{4}a$, and a; and $y = z = 0$. From (19) and (20) we get by means of the preceding data the following table of values for use in (21):

$\frac{x}{a}$	v_0'	S_1'	$3S_2'$	$-t\frac{\partial S_2'}{\partial t}$	$-x\frac{\partial S_2'}{\partial x}$
0	1.0047	+ 0.0828	+ 0.0255	+ 0.0075	+ 0.0000
$\frac{1}{4}$	1.0028	+ 0.0596	+ 0.0181	+ 0.0053	+ 0.0170
$\frac{1}{2}$	0.9970	+ 0.0015	+ 0.0001	+ 0.0000	+ 0.0268
$\frac{3}{4}$	0.9875	− 0.0596	− 0.0181	− 0.0053	+ 0.0208
1	0.9741	− 0.0859	− 0.0257	− 0.0075	+ 0.0000

In addition we have

$$v_0'' = v_0''' = 0.87054,$$

$$S_1'' = S_1''' = +0.0828,$$

and

$$\tfrac{1}{2}\frac{\partial^2 V''}{\partial \eta^2} = \tfrac{1}{2}\frac{\partial^2 V'''}{\partial \zeta^2} = +0.0330.$$

Hence we find the following table of values of V', in which the terms in ξ and ξ^2 are given separately in order to show their magnitudes relatively to v_0':

$\frac{x}{a}$	v_0'	$\xi\frac{\partial V'}{\partial \xi}$	$\frac{1}{2}\xi^2\frac{\partial^2 V'}{\partial \xi^2}$	V'
0	+ 1.0047	− 0.0052	+ 0.00013	= 0.9996
$\frac{1}{4}$	+ 1.0028	− 0.0037	+ 0.00016	= 0.9993
$\frac{1}{2}$	+ 0.9970	− 0.0001	+ 0.00010	= 0.9970
$\frac{3}{4}$	+ 0.9875	+ 0.0037	− 0.00001	= 0.9912
1	+ 0.9741	+ 0.0054	− 0.00013	= 0.9794

We find also that

$$V'' = V''' = 0.87054 - 0.000207 + 0.0000002$$
$$= 0.87033.$$

It is now only necessary to form the products $V' V'' V'''$ in order to get the temperature u in terms of u_0. The results are

$t = 20^m, y = z = 0.$	
$\frac{x}{a}$	$\frac{u}{u_0}$
0	0.7571
$\frac{1}{4}$	0.7570
$\frac{1}{2}$	0.7553
$\frac{3}{4}$	0.7509
1	0.7420

It appears from the values in the second paragraph above that the terms in ξ^2, η^2, and ζ^2 are, for the assumed values of those ratios, already practically insignificant at the end of twenty minutes from the initial epoch. Hence in computing temperatures for greater values of the time we may properly neglect all terms of the second order. For the purpose of showing the march of the temperature with the time, we have computed the values which u has at the above named points along the axis of the bar for intervals of twenty minutes up to $t = 5^h\ 40^m$. Since minute precision is not needed in this work, we have used $V' = v_0' - \xi S_1'$ and $V'' = V''' = v_0''$. The resulting values are shown in the table below. The average temperature of the whole mass of the bar obtained from (22) is also given in the last column of the table. This average is only slightly greater ($\frac{1}{1200}$th part) than the average temperature of the face $2a \times 2b$.*

*The question whether under similar circumstances there can be any detrimental internal strain due to inequalities in distribution of temperature in standards of length of ordinary forms and dimensions, appears to receive a decisively negative answer in the computed values of the table. Such standards are generally smaller in cross section than our assumed bar, and would, therefore, other things being equal, sooner attain a sensibly uniform temperature. Hence we infer that, so far as flexure due to temperature is concerned, it is immaterial where the graduations on a standard are placed—whether on its external surface or on its neutral plane.

Time t	Values of temperature u on axis of bar where $x/a=$					Average for mass of bar.
	o	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	
h m						
0 20	$0.757u_0$	$0.757u_0$	$0.755u_0$	$0.751u_0$	$0.742u_0$	$0.754u_0$
0 40	0.573	0.572	0.570	0.565	0.558	0.568
1 00	0.433	0.432	0.430	0.426	0.420	0.428
1 20	0.326	0.326	0.324	0.321	0.317	0.323
1 40	0.246	0.246	0.244	0.242	0.239	0.243
2 00	0.185	0.185	0.184	0.182	0.180	0.183
2 20	0.140	0.140	0.139	0.137	0.136	0.138
2 40	0.105	0.105	0.104	0.103	0.102	0.104
3 00	0.079	0.079	0.079	0.078	0.077	0.079
3 20	0.060	0.060	0.059	0.059	0.058	0.059
3 40	0.045	0.045	0.045	0.044	0.044	0.045
4 00	0.034	0.034	0.034	0.033	0.033	0.034
4 20	0.026	0.026	0.025	0.025	0.025	0.025
4 40	0.019	0.019	0.019	0.019	0.019	0.019
5 00	0.015	0.014	0.014	0.014	0.014	0.014
5 20	0.011	0.011	0.011	0.011	0.011	0.011
5 40	0.008	0.008	0.008	0.008	0.008	0.008

12. Turning now to the question of numerical applications for the case in which ξ, η, and ζ exceed unity, it will be advantageous to first set down some additional formulas for computing the differential coefficients which appear in the second members of (25) and (26).

To avoid writing frequently recurring complex expressions, let us make the following abbreviations:

$$
\left.
\begin{aligned}
X_1 &= \sum_{n=0}^{n=\infty} (-1)^n (2n+1) e^{-(\pi/4r_0)^2 (2n+1)^2} \cos \tfrac{1}{2}(2n+1)\pi a^{-1} x, \\
X_2 &= \sum_{n=0}^{n=\infty} (-1)^n e^{-(\pi/4r_0)^2 (2n+1)^2} \sin \tfrac{1}{2}(2n+1)\pi a^{-1} x, \\
X_3 &= \sum_{n=0}^{n=\infty} (-1)^n (2n+1)^3 e^{-(\pi/4r_0)^2 (2n+1)^2} \cos \tfrac{1}{2}(2n+1)\pi a^{-1} x, \\
X_4 &= \sum_{n=0}^{n=\infty} (-1)^n (2n+1)^2 e^{-(\pi/4r_0)^2 (2n+1)^2} \sin \tfrac{1}{2}(2n+1)\pi a^{-1} x,
\end{aligned}
\right\} \quad (41)
$$

$$r_0^2 = a^2/4kt.$$

Then, the second and third of (25) become

$$\begin{aligned}\frac{\partial V'}{\partial \xi_1} &= +\tfrac{1}{2}\pi r_0^{-2} X_1 + 2a^{-1}xX_2,\\ \frac{\partial^2 V}{\partial \xi_1^2} &= -\pi(1 + a^{-2}x^2 + \tfrac{3}{2}r_0^{-2})X_1 + \tfrac{1}{16}\pi^3 r_0^{-4} X_3\\ &\quad - 4a^{-1}xX_2 + \tfrac{1}{2}\pi^2 r_0^{-2} a^{-1}xX_4.\end{aligned}\tag{42}$$

The equations just given will be useful when r_0 is small, since the functions in (41) will then converge very rapidly. For the converse case, we must make use of the differential coefficients given by (35). Introducing these in the second and third of (25), we get

$$\begin{aligned}\frac{\partial V'}{\partial \xi_1} &= +\frac{2}{\sqrt{\pi}}(rR_1 + R_2)\\ &= +\frac{2r_0}{\sqrt{\pi}}\left[e^{-(r_0-r)^2} + e^{-(r_0+r)^2} - 3e^{-(3r_0-r)^2} - \ldots\right],\\ \frac{\partial^2 V'}{\partial \xi_1^2} &= -\frac{4}{\sqrt{\pi}}\left[(r_0^2 + r^2)R_2 + 2rR_3 + R_4\right].\end{aligned}\tag{43}$$

$$r^2 = \frac{x^2}{4kt}$$

The last expression assumes very easily managed forms for certain values of r used in the sequel. For the benefit of the reader who may desire to verify our work, these forms are given below:

$$\begin{aligned}\frac{\partial^2 V'}{\partial \xi_1^2} &= -16\frac{r_0^3}{\sqrt{\pi}}\left(e^{-r_0^2} - 15e^{-9r_0^2} + 65e^{-25r_0^2} - \ldots\right) \text{ for } r = 0,\\ &= -4\frac{r_0^3}{\sqrt{\pi}}\left(e^{-(\frac{1}{2}r_0)^2} + 3e^{-9(\frac{1}{2}r_0)^2} - 25e^{-25(\frac{1}{2}r_0)^2} - \ldots\right) \text{ for } r = \tfrac{1}{2}r_0,\\ &= +64\frac{r_0^3}{\sqrt{\pi}}\left(e^{-4r_0^2} - 4e^{-16r_0^2} + 9e^{-36r_0^2} - \ldots\right) \text{ for } r = r_0.\end{aligned}\tag{44}$$

The series in (43) and (44) converge with extreme suddenness for large values of r_0, and they cease to be applicable only when r_0 is very small.

With preliminaries thus arranged, let us consider the behavior of a bar of the same dimensions and material as the bar of the previous example, cooling in a medium which produces an emissivity great enough to make ξ, η, and ζ exceed unity. Without assigning to these symbols any particular values, it appears that the bar would very quickly assume the temperature of the medium. For we have, as in § 11,

$$a = 50^{cm},\ b = c = 2^{cm},$$

and

$$k = 0.185;$$

and since for the transverse section of the bar

$$r_0^{\,2} = \frac{b^2}{4kt} = \frac{1}{kt},$$

the quantities V_0'', V_0''' and the derivatives of V'' and V''' relatively to η^{-1} and ζ^{-1} become practically *nil* soon after the initial epoch. Thus, from (25) and (42), we find the following table of values which show the rapid decrease of those quantities with the time

t	V_0''	$\dfrac{\partial V''}{\partial \eta^{-1}}$	$\dfrac{\partial^2 V''}{\partial \eta^{-2}}$
s 30	0.041	+ 0.284	+ 0.991
60	0.0013	+ 0.018	+ 0.195
90	0.00004	+ 0.0009	+ 0.016

Hence we see that, with such values of ξ, η, and ζ as render equations (26) applicable, the bar in question would have sensibly the same temperature as the surrounding medium in less than two minutes after the initial epoch.

Other conditions remaining constant, the rate of cooling of a mass diminishes very rapidly with an increase of its dimensions. In order to strikingly illustrate this fact by means of the analysis, we may contrast the cooling of the bar just considered with the cooling of a cubic meter of the same material. The use of the formulas will be sufficiently indicated by computing the temperatures, corresponding to a few values of the time, at the five points designated by the following values of their co-ordinates:

x	y	z
0	0	0
$\pm \frac{1}{2}a$	0	0
$\pm\ a$	0	0

To compute V', V'', V''' for such values of the time as we have assumed, one may use equations (25) and (42) or equations (32) and (43), the latter being best for the smaller and the former best for the greater times. We have used

both sets of formulas as a check on the computed results. The quantities needed in equations (26) are given in the following table. V_0', V_0'', and V_0''' and the corresponding derivatives of V', V'', and V''' are equal for $x = y = z = 0$. Hence the values in the first section of the table are those required by the second and third of (26).

t	$x=0,\ y=0,\ z=0.$			$x=\pm\frac{1}{2}a,\ y=0,\ z=0.$		
	V_0'	$\frac{\partial V'}{\partial \xi_1}$	$\frac{\partial^2 V'}{\partial \xi_1^2}$	V_0'	$\frac{\partial V'}{\partial \xi_1}$	$\frac{\partial^2 V'}{\partial \xi_1^2}$
h m						
0 20	0.965	+ 0.227	− 2.55	0.764	+ 0.940	− 5.33
1 00	0.659	+ 0.852	− 3.50	0.467	+ 0.874	− 1.85
2 00	0.342	+ 0.905	− 4.13	0.242	+ 0.825	− 1.11
3 00	0.177	+ 0.699	+ 1.26	0.125	+ 0.587	+ 3.80

t	$x=\pm a,\ y=0,\ z=0.$		
	V_0'	$\frac{\partial V'}{\partial \xi_1}$	$\frac{\partial^2 V'}{\partial \xi_1^2}$
h m			
0 20	0	+ 1.893	+ 0.00
1 00	0	+ 1.042	+ 0.84
2 00	0	+ 0.537	+ 5.45
3 00	0	+ 0.278	+ 9.37

Let us now suppose that $\xi = \eta = \zeta = 10$; *i. e.*, $ah = 50h = 10$, or $h = H/K = \frac{1}{5}$. Then we have for the centre of the cube when $t = 20^m$,

$$V' = V'' = V''' = 0.965 + 0.023 - 0.013 + T_3$$
$$= 0.975,$$

and
$$u = (0.975)^3 u_0 = 0.927\ u_0.$$

For a point on either axis of the cube half way between the centre and a face, we have when $t = 1^h$,

$$V' = 0.467 + 0.087 - 0.009 + T_3,$$
$$V'' = V''' = 0.659 + 0.085 - 0.018 + T_3;$$

and hence

$$u = 0.545 \times (0.726)^2 u_0 = 0.287\ u_0.$$

In a similar manner the temperature at the centre of a face of the cube is found. The following table gives all of the temperatures derivable from the data in the preceding table. The values of the co-ordinates may, of course, be interchanged, so that the temperatures given in the last two columns are common respectively to six points.

Time.	Temperature.		
t	At centre of cube.	At middle point of semi axis.	At centre of face.
h m			
0 20	$0.927u_0$	$0.790u_0$	$0.180u_0$
1 00	0.384	0.288	0.057
2 00	0.069	0.054	0.014
3 00	0.016	0.013	0.005

As a final example in this section we may give the average values of the temperature of the whole mass and of any face of the cube just considered. These values are easily computed from equations (36) to (40). Using the same values of the time as those in the last table above, we arrive at the following results, exhibited also in tabular form. The quantities N and N_1 are computed from (39); M' from (38), and V_c''' comes from the data in the third section of the second table above. It is assumed that $\xi = \eta = \zeta = 10$.

t	N	N_1	M'	V_c'''	Average temperature	
					Of whole mass.	Of any face.
h m						
0 20	0.689	+ 1.34	0.823	0.189	$0.56u_0$	$0.13u_0$
1 00	0.422	+ 1.53	0.575	0.113	0.19	0.04
2 00	0.331	+ 1.37	0.468	0.108	0.10	0.02
3 00	0.113	+ 0.99	0.212	0.122	0.01	0.01

13. It appears desirable to indicate in the last section of this paper in what ways the conditions of the problem considered and the analysis derived may be useful in determining the thermal constants of a bar; namely, the thermal capacity of C, the conductivity K, and the emissivity H.

The principal conditions of the problem we have treated are first, that the bar has initially a uniform temperature, and second, that it cools or heats in a medium whose temperature remains sensibly constant. The facts most readily observed under these conditions are the initial excess in temperature of the bar above that of the medium and the variation of its temperature or some function thereof with the time. From a sufficient number of such facts it would appear that all of the constants might be determined. To fix the ideas, suppose the excess in temperature at the centre of the bar or the average excess of its entire mass, can be measured at any time. Call this excess u'. Then we have

$$u' = f(C, H, K);$$

and if we write

$$C_0 + \Delta C = C,$$
$$H_0 + \Delta H = H,$$
$$K_0 + \Delta K = K,$$

in which C_0, H_0, and K_0 are approximate values of C, H, and K found in any manner, we shall have the following observation-equation:

$$\frac{\partial u'}{\partial C}\Delta C + \frac{\partial u'}{\partial H}\Delta H + \frac{\partial u'}{\partial K}\Delta K + f(C_0, H_0, K_0) - u' = \Delta u'. \tag{45}$$

From a series of such equations, differing from each other sufficiently, one could by the method of least squares derive precise corrections to the approximate values C_0, H_0, and K_0. The applicability of this process, however, is limited narrowly by the difficulty of securing a sufficient range in temperature in a given bar, or by the difficulty of securing a sufficient range of dimensions in different bars or masses of a given material.

Although it does not appear practicable under the special conditions of the present problem to derive all the thermal constants of a bar, there exist between those constants under certain circumstances some relations which may be accurately determined. These we proceed to point out.

Firstly, when the emissivity and the dimensions of the bar are small, we have from (9) and (22), the average temperature of the bar's mass

$$u' = u_0 e^{-kt(\lambda_0^2 + \mu_0^2 + \nu_0^2)} + T_2,$$

whence

$$k(\lambda_0^2 + \mu_0^2 + \nu_0^2) \log e = t^{-1} \log \frac{u_0}{u'}$$

But

$$k = \frac{K}{C},$$

and from (9) and (12)

$$\lambda_0^2 + \mu_0^2 + \nu_0^2 = (a^{-1} + b^{-1} + c^{-1}) \frac{H}{K} + T_2.$$

Hence, to terms of the first order in H we have

$$(a^{-1} + b^{-1} + c^{-1}) \frac{H}{C} \log e = t^{-1} \log \frac{u_0}{u'}. \tag{46}$$

This expression gives the emissivity H in terms of the thermal capacity C when the second member is known by observation. Under the assumed condition that H is small, u' may be safely inferred from the length of the bar (supposing it to be of small cross section). In case there are several observed values of u' and the corresponding time t, let W be the most probable value of the second member of (46). Then, since u' or $\log u'$ only can properly be considered subject to error of observation, we have in the usual notation of least squares

$$W = \frac{\left[t \log \frac{u_0}{u'}\right]}{[t^2]} \tag{47}$$

From this and (46) therefore,

$$H = \frac{CW(a^{-1} + b^{-1} + c^{-1})^{-1}}{\log e}. \tag{48}$$

Secondly, when the emissivity is large or when ξ, η and ζ are such, we have from (6) and (26),

$$\frac{u}{u_0} = V_0' V_0'' V_0''' \left[1 + \left(a^{-1} V_0'^{-1} \frac{\partial V'}{\partial \xi^{-1}} + \dots \right) \frac{K}{H} + T_2\right]. \tag{49}$$

Now V_0', V_0'', V_0''' and the derivatives of V', V'', V''' involve the coefficient of diffusion or C and K only; and hence when these are known an observation of u and t will give H through the relation (49). On the other hand, if ξ, η, and ζ are so large that the terms beyond the first in (49) may be neglected, an observation of u and t will give the coefficient of diffusion k. The method of treating a series of observed values of u and t to determine H or k, is sufficiently indicated by the remarks pertaining to equations (45) and (47).